Kenneth H. Rosen

Discrete Mathematics and Its Applications

Eighth Edition

Mc Graw Hill Education

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Kenneth H. Rosen
formerly AT&T Laboratories

McGraw Hill Education

DISCRETE MATHEMATICS AND ITS APPLICATIONS, EIGHTH EDITION

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Contents

About the Author

Kenneth H. Rosen received his B.S. in Mathematics from the University of Michigan, Ann Arbor (1972), and his Ph.D. in Mathematics from M.I.T. (1976), where he wrote his thesis in number theory under the direction of Harold Stark. Before joining Bell Laboratories in 1982, he held positions at the University of Colorado, Boulder; The Ohio State University, Columbus; and the University of Maine, Orono, where he was an associate professor of mathematics. He enjoyed a long career as a Distinguished Member of the Technical Staff at AT&T Bell Laboratories (and AT&T Laboratories) in Monmouth County, New Jersey. While working at Bell Labs, he taught at Monmouth University, teaching courses in discrete mathematics, coding theory, and data security. After leaving AT&T Labs, he became a visiting research professor of computer science at Monmouth University, where he has taught courses in algorithm design, computer security and cryptography, and discrete mathematics.

Dr. Rosen has published numerous articles in professional journals on number theory and on mathematical modeling. He is the author of the widely used *Elementary Number Theory and Its Applications*, published by Pearson, currently in its sixth edition, which has been translated into Chinese. He is also the author of *Discrete Mathematics and Its Applications*, published by McGraw-Hill, currently in its eighth edition. *Discrete Mathematics and Its Applications* has sold more than 450,000 copies in North America during its lifetime, and hundreds of thousands of copies throughout the rest of the world. This book has also been translated into many languages, including Spanish, French, Portuguese, Greek, Chinese, Vietnamese, and Korean. He is also co-author of *UNIX: The Complete Reference*; *UNIX System V Release 4: An Introduction*; and *Best UNIX Tips Ever*, all published by Osborne McGraw-Hill. These books have sold more than 150,000 copies, with translations into Chinese, German, Spanish, and Italian. Dr. Rosen is also the editor of both the first and second editions (published in 1999 and 2018, respectively) of the *Handbook of Discrete and Combinatorial Mathematics*, published by CRC Press. He has served as the advisory editor of the CRC series of books in discrete mathematics, sponsoring more than 70 volumes on diverse aspects of discrete mathematics, many of which are introduced in this book. He is an advisory editor for the CRC series of mathematics textbooks, where he has helped more than 30 authors write better texts. Dr. Rosen serves as an Associate Editor for the journal *Discrete Mathematics*, where he handles papers in many areas, including graph theory, enumeration, number theory, and cryptography.

Dr. Rosen has had a longstanding interest in integrating mathematical software into the educational and professional environments. He has worked on several projects with Waterloo Maple Inc.'s Maple™ software in both these areas. Dr. Rosen has devoted a great deal of energy to ensuring that the online homework for *Discrete Mathematics and its Applications* is a superior teaching tool. Dr. Rosen has also worked with several publishing companies on their homework delivery platforms.

At Bell Laboratories and AT&T Laboratories, Dr. Rosen worked on a wide range of projects, including operations research studies, product line planning for computers and data communications equipment, technology assessment and innovation, and many other efforts. He helped plan AT&T's products and services in the area of multimedia, including video communications, speech recognition, speech synthesis, and image networking. He evaluated new technology for use by AT&T and did standards work in the area of image networking. He also invented many new services, and holds more than 70 patents. One of his more interesting projects involved helping evaluate technology for the AT&T attraction that was part of EPCOT Center. After leaving AT&T, Dr. Rosen has worked as a technology consultant for Google and for AT&T.

Preface

In writing this book, I was guided by my long-standing experience and interest in teaching discrete mathematics. For the student, my purpose was to present material in a precise, readable manner, with the concepts and techniques of discrete mathematics clearly presented and demonstrated. My goal was to show the relevance and practicality of discrete mathematics to students, who are often skeptical. I wanted to give students studying computer science all of the mathematical foundations they need for their future studies. I wanted to give mathematics students an understanding of important mathematical concepts together with a sense of why these concepts are important for applications. And most importantly, I wanted to accomplish these goals without watering down the material.

For the instructor, my purpose was to design a flexible, comprehensive teaching tool using proven pedagogical techniques in mathematics. I wanted to provide instructors with a package of materials that they could use to teach discrete mathematics effectively and efficiently in the most appropriate manner for their particular set of students. I hope that I have achieved these goals.

I have been extremely gratified by the tremendous success of this text, including its use by more than one million students around the world over the last 30 years and its translation into many different languages. The many improvements in the eighth edition have been made possible by the feedback and suggestions of a large number of instructors and students at many of the more than 600 North American schools, and at many universities in different parts of the world, where this book has been successfully used. I have been able to significantly improve the appeal and effectiveness of this book edition to edition because of the feedback I have received and the significant investments that have been made in the evolution of the book.

This text is designed for a one- or two-term introductory discrete mathematics course taken by students in a wide variety of majors, including mathematics, computer science, and engineering. College algebra is the only explicit prerequisite, although a certain degree of mathematical maturity is needed to study discrete mathematics in a meaningful way. This book has been designed to meet the needs of almost all types of introductory discrete mathematics courses. It is highly flexible and extremely comprehensive. The book is designed not only to be a successful textbook, but also to serve as a valuable resource students can consult throughout their studies and professional life.

Goals of a Discrete Mathematics Course

A discrete mathematics course has more than one purpose. Students should learn a particular set of mathematical facts and how to apply them; more importantly, such a course should teach students how to think logically and mathematically. To achieve these goals, this text stresses mathematical reasoning and the different ways problems are solved. Five important themes are interwoven in this text: mathematical reasoning, combinatorial analysis, discrete structures, algorithmic thinking, and applications and modeling. A successful discrete mathematics course should carefully blend and balance all five themes.

1. *Mathematical Reasoning:* Students must understand mathematical reasoning in order to read, comprehend, and construct mathematical arguments. This text starts with a discussion of mathematical logic, which serves as the foundation for the subsequent discussions of methods of proof. Both the science and the art of constructing proofs are addressed. The technique of

mathematical induction is stressed through many different types of examples of such proofs and a careful explanation of why mathematical induction is a valid proof technique.

2. *Combinatorial Analysis:* An important problem-solving skill is the ability to count or enumerate objects. The discussion of enumeration in this book begins with the basic techniques of counting. The stress is on performing combinatorial analysis to solve counting problems and analyze algorithms, not on applying formulae.
3. *Discrete Structures:* A course in discrete mathematics should teach students how to work with discrete structures, which are the abstract mathematical structures used to represent discrete objects and relationships between these objects. These discrete structures include sets, permutations, relations, graphs, trees, and finite-state machines.
4. *Algorithmic Thinking:* Certain classes of problems are solved by the specification of an algorithm. After an algorithm has been described, a computer program can be constructed implementing it. The mathematical portions of this activity, which include the specification of the algorithm, the verification that it works properly, and the analysis of the computer memory and time required to perform it, are all covered in this text. Algorithms are described using both English and an easily understood form of pseudocode.
5. *Applications and Modeling:* Discrete mathematics has applications to almost every conceivable area of study. There are many applications to computer science and data networking in this text, as well as applications to such diverse areas as chemistry, biology, linguistics, geography, business, and the Internet. These applications are natural and important uses of discrete mathematics and are not contrived. Modeling with discrete mathematics is an extremely important problem-solving skill, which students have the opportunity to develop by constructing their own models in some of the exercises.

Changes in the Eighth Edition

Although the seventh edition has been an extremely effective text, many instructors have requested changes to make the book more useful to them. I have devoted a significant amount of time and energy to satisfy their requests and I have worked hard to find my own ways to improve the book and to keep it up-to-date.

The eighth edition includes changes based on input from more than 20 formal reviewers, feedback from students and instructors, and my insights. The result is a new edition that I expect will be a more effective teaching tool. Numerous changes in the eighth edition have been designed to help students learn the material. Additional explanations and examples have been added to clarify material where students have had difficulty. New exercises, both routine and challenging, have been added. Highly relevant applications, including many related to the Internet, to computer science, and to mathematical biology, have been added. The companion website has benefited from extensive development; it now provides extensive tools students can use to master key concepts and to explore the world of discrete mathematics. Furthermore, additional effective and comprehensive learning and assessment tools are available, complementing the textbook.

I hope that instructors will closely examine this new edition to discover how it might meet their needs. Although it is impractical to list all the changes in this edition, a brief list that highlights some key changes, listed by the benefits they provide, may be useful.

Changes in the Eighth Edition

This new edition of the book includes many enhancements, updates, additions, and edits, all designed to make the book a more effective teaching tool for a modern discrete mathematics course. Instructors who have used the book previously will notice overall changes that have been made throughout the book, as well as specific changes. The most notable revisions are described here.

Overall Changes

- Exposition has been improved throughout the book with a focus on providing more clarity to help students read and comprehend concepts.
- Many proofs have been enhanced by adding more details and explanations, and by reminding the reader of the proof methods used.
- New examples have been added, often to meet needs identified by reviewers or to illustrate new material. Many of these examples are found in the text, but others are available only on the companion website.
- Many new exercises, both routine and challenging, address needs identified by instructors or cover new material, while others strengthen and broaden existing exercise sets.
- More second and third level heads have been used to break sections into smaller coherent parts, and a new numbering scheme has been used to identify subsections of the book.
- The online resources for this book have been greatly expanded, providing extensive support for both instructors and students. These resources are described later in the front matter.

Topic Coverage

- **Logic** Several logical puzzles have been introduced. A new example explains how to model the n-queens problem as a satisfiability problem that is both concise and accessible to students.
- **Set theory** Multisets are now covered in the text. (Previously they were introduced in the exercises.)
- **Algorithms** The string matching problem, an important algorithm for many applications, including spell checking, key-word searching, string-matching, and computational biology, is now discussed. The brute-force algorithm for solving string-matching exercises is presented.
- **Number theory** The new edition includes the latest numerical and theoretic discoveries relating to primes and open conjectures about them. The extended Euclidean algorithm, a one-pass algorithm, is now discussed in the text. (Previously it was covered in the exercises.)
- **Cryptography** The concept of homomorphic encryption, and its importance to cloud computing, is now covered.
- **Mathematical induction** The template for proofs by mathematical induction has been expanded. It is now placed in the text before examples of proof by mathematical induction.
- **Counting methods** The coverage of the division rule for counting has been expanded.
- **Data mining** Association rules—key concepts in data mining—are now discussed in the section on n-ary relations. Also, the Jaccard metric, which is used to find the distance between two sets and which is used in data mining, is introduced in the exercises.
- **Graph theory applications** A new example illustrates how semantic networks, an important structure in artificial intelligence, can be modeled using graphs.

- **Biographies** New biographies of Wiles, Bhaskaracharya, de la Vallée-Poussin, Hadamard, Zhang, and Gentry have been added. Existing biographies have been expanded and updated. This adds diversity by including more historically important Eastern mathematicians, major nineteenth and twentieth century researchers, and currently active twenty-first century mathematicians and computer scientists.

Features of the Book

ACCESSIBILITY This text has proven to be easily read and understood by many beginning students. There are no mathematical prerequisites beyond college algebra for almost all the contents of the text. Students needing extra help will find tools on the companion website for bringing their mathematical maturity up to the level of the text. The few places in the book where calculus is referred to are explicitly noted. Most students should easily understand the pseudocode used in the text to express algorithms, regardless of whether they have formally studied programming languages. There is no formal computer science prerequisite.

Each chapter begins at an easily understood and accessible level. Once basic mathematical concepts have been carefully developed, more difficult material and applications to other areas of study are presented.

FLEXIBILITY This text has been carefully designed for flexible use. The dependence of chapters on previous material has been minimized. Each chapter is divided into sections of approximately the same length, and each section is divided into subsections that form natural blocks of material for teaching. Instructors can easily pace their lectures using these blocks.

WRITING STYLE The writing style in this book is direct and pragmatic. Precise mathematical language is used without excessive formalism and abstraction. Care has been taken to balance the mix of notation and words in mathematical statements.

MATHEMATICAL RIGOR AND PRECISION All definitions and theorems in this text are stated extremely carefully so that students will appreciate the precision of language and rigor needed in mathematics. Proofs are motivated and developed slowly; their steps are all carefully justified. The axioms used in proofs and the basic properties that follow from them are explicitly described in an appendix, giving students a clear idea of what they can assume in a proof. Recursive definitions are explained and used extensively.

WORKED EXAMPLES Over 800 examples are used to illustrate concepts, relate different topics, and introduce applications. In most examples, a question is first posed, then its solution is presented with the appropriate amount of detail.

APPLICATIONS The applications included in this text demonstrate the utility of discrete mathematics in the solution of real-world problems. This text includes applications to a wide variety of areas, including computer science, data networking, psychology, chemistry, engineering, linguistics, biology, business, and the Internet.

ALGORITHMS Results in discrete mathematics are often expressed in terms of algorithms; hence, key algorithms are introduced in most chapters of the book. These algorithms are expressed in words and in an easily understood form of structured pseudocode, which is described and specified in Appendix 3. The computational complexity of the algorithms in the text is also analyzed at an elementary level.

HISTORICAL INFORMATION The background of many topics is succinctly described in the text. Brief biographies of 89 mathematicians and computer scientists are included as footnotes. These biographies include information about the lives, careers, and accomplishments of these important contributors to discrete mathematics, and images, when available, are displayed.

In addition, numerous historical footnotes are included that supplement the historical information in the main body of the text. Efforts have been made to keep the book up-to-date by reflecting the latest discoveries.

KEY TERMS AND RESULTS A list of key terms and results follows each chapter. The key terms include only the most important that students should learn, and not every term defined in the chapter.

EXERCISES There are over 4200 exercises in the text, with many different types of questions posed. There is an ample supply of straightforward exercises that develop basic skills, a large number of intermediate exercises, and many challenging exercises. Exercises are stated clearly and unambiguously, and all are carefully graded for level of difficulty. Exercise sets contain special discussions that develop new concepts not covered in the text, enabling students to discover new ideas through their own work.

Exercises that are somewhat more difficult than average are marked with a single star, *; those that are much more challenging are marked with two stars, **. Exercises whose solutions require calculus are explicitly noted. Exercises that develop results used in the text are clearly identified with the right pointing hand symbol, ☞. Answers or outlined solutions to all odd-numbered exercises are provided at the back of the text. The solutions include proofs in which most of the steps are clearly spelled out.

REVIEW QUESTIONS A set of review questions is provided at the end of each chapter. These questions are designed to help students focus their study on the most important concepts and techniques of that chapter. To answer these questions students need to write long answers, rather than just perform calculations or give short replies.

SUPPLEMENTARY EXERCISE SETS Each chapter is followed by a rich and varied set of supplementary exercises. These exercises are generally more difficult than those in the exercise sets following the sections. The supplementary exercises reinforce the concepts of the chapter and integrate different topics more effectively.

COMPUTER PROJECTS Each chapter is followed by a set of computer projects. The approximately 150 computer projects tie together what students may have learned in computing and in discrete mathematics. Computer projects that are more difficult than average, from both a mathematical and a programming point of view, are marked with a star, and those that are extremely challenging are marked with two stars.

COMPUTATIONS AND EXPLORATIONS A set of computations and explorations is included at the conclusion of each chapter. These exercises (approximately 120 in total) are designed to be completed using existing software tools, such as programs that students or instructors have written or mathematical computation packages such as Maple™ or Mathematica™. Many of these exercises give students the opportunity to uncover new facts and ideas through computation. (Some of these exercises are discussed in the *Exploring Discrete Mathematics* companion workbooks available online.)

WRITING PROJECTS Each chapter is followed by a set of writing projects. To do these projects students need to consult the mathematical literature. Some of these projects are historical in nature and may involve looking up original sources. Others are designed to serve as gateways to new topics and ideas. All are designed to expose students to ideas not covered in depth in the text. These projects tie mathematical concepts together with the writing process and help expose students to possible areas for future study. (Suggested references for these projects can be found online or in the printed *Student's Solutions Guide*.)

APPENDICES There are three appendices to the text. The first introduces axioms for real numbers and the positive integers, and illustrates how facts are proved directly from these axioms. The second covers exponential and logarithmic functions, reviewing some basic material used heavily in the course. The third specifies the pseudocode used to describe algorithms in this text.

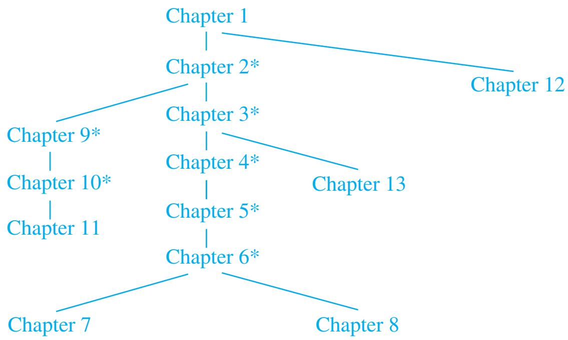
SUGGESTED READINGS A list of suggested readings for the overall book and for each chapter is provided after the appendices. These suggested readings include books at or below the level of this text, more difficult books, expository articles, and articles in which discoveries in discrete mathematics were originally published. Some of these publications are classics, published many years ago, while others have been published in the last few years. These suggested readings are complemented by the many links to valuable resources available on the web that can be found on the website for this book.

How to Use This Book

This text has been carefully written and constructed to support discrete mathematics courses at several levels and with differing foci. The following table identifies the core and optional sections. An introductory one-term course in discrete mathematics at the sophomore level can be based on the core sections of the text, with other sections covered at the discretion of the instructor. A two-term introductory course can include all the optional mathematics sections in addition to the core sections. A course with a strong computer science emphasis can be taught by covering some or all of the optional computer science sections. Instructors can find sample syllabi for a wide range of discrete mathematics courses and teaching suggestions for using each section of the text can be found in the *Instructor's Resource Guide* available on the website for this book.

Chapter	*Core*	*Optional CS*	*Optional Math*
1	1.1–1.8 (as needed)		
2	2.1–2.4, 2.6 (as needed)		2.5
3		3.1–3.3 (as needed)	
4	4.1–4.4 (as needed)	4.5, 4.6	
5	5.1–5.3	5.4, 5.5	
6	6.1–6.3	6.6	6.4, 6.5
7	7.1	7.4	7.2, 7.3
8	8.1, 8.5	8.3	8.2, 8.4, 8.6
9	9.1, 9.3, 9.5	9.2	9.4, 9.6
10	10.1–10.5		10.6–10.8
11	11.1	11.2, 11.3	11.4, 11.5
12		12.1–12.4	
13		13.1–13.5	

Instructors using this book can adjust the level of difficulty of their course by choosing either to cover or to omit the more challenging examples at the end of sections, as well as the more challenging exercises. The chapter dependency chart shown here displays the strong dependencies. A star indicates that only relevant sections of the chapter are needed for study of a later chapter. Weak dependencies have been ignored. More details can be found in the *Instructor's Resource Guide*.



Ancillaries

STUDENT'S SOLUTIONS GUIDE This student manual, available separately, contains full solutions to all odd-numbered exercises in the exercise sets. These solutions explain why a particular method is used and why it works. For some exercises, one or two other possible approaches are described to show that a problem can be solved in several different ways. Suggested references for the writing projects found at the end of each chapter are also included in this volume. Also included are a guide to writing proofs and an extensive description of common mistakes students make in discrete mathematics, plus sample tests and a sample crib sheet for each chapter designed to help students prepare for exams.

INSTRUCTOR'S RESOURCE GUIDE This manual, available on the website and in printed form by request for instructors, contains full solutions to even-numbered exercises in the text. Suggestions on how to teach the material in each chapter of the book are provided, including the points to stress in each section and how to put the material into perspective. It also offers sample tests for each chapter and a test bank containing over 1500 exam questions to choose from. Answers to all sample tests and test bank questions are included. Finally, sample syllabi are presented for courses with differing emphases and student ability levels.

Acknowledgments

I would like to thank the many instructors and students at a variety of schools who have used this book and provided me with their valuable feedback and helpful suggestions. Their input has made this a much better book than it would have been otherwise. I especially want to thank Jerrold Grossman and Dan Jordan for their technical reviews of the eighth edition and their "eagle eyes," which have helped ensure the accuracy and quality of this book. Both have proofread every part of the book many times as it has gone through the different steps of production and have helped eliminate old errata and prevented the insertion of new errata.

Thanks go to Dan Jordan for his work on the student solutions manual and instructor's resource guide. He has done an admirable job updating these ancillaries. Jerrold Grossman, the author of these ancillaries for the first seven editions of the book, has provided valuable assistance to Dan. I would also like to express my gratitude to the many people who have helped create and maintain the online resources for this book. In particular, special thanks go to Dan Jordan and Rochus Boerner for their extensive work improving online questions for the Connect Site, described later in this preface.

I thank the reviewers of this eighth and all previous editions. These reviewers have provided much helpful criticism and encouragement to me. I hope this edition lives up to their high expectations. There have been well in excess of 200 reviewers of this book since its first edition, with many from countries other than the United States. The most recent reviewers are listed here.

Recent Reviewers

Barbara Anthony
Southwestern University

Philip Barry
University of Minnesota, Minneapolis

Benkam Bobga
University of North Georgia

Miklos Bona
University of Florida

Steve Brick
University of South Alabama

Kirby Brown
Queens College

John Carter
University of Toronto

Narendra Chaudhari
Nanyang Technological University

Tim Chappell
Penn Valley Community College

Allan Cochran
University of Arkansas

Daniel Cunningham
Buffalo State College

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Georgia State University

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The Ohio State University

Ronald Dotzel
University of Missouri-St. Louis

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Columbus State Community College

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University of Michigan, Dearborn

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Purdue University,
Calumet-Hammond

Herbert Enderton
University of California, Los Angeles

Anthony Evans
Wright State University

Kim Factor
Marquette University

Margaret Fleck
University of Illinois, Champaign

Melissa Gaddini
Robert Morris University

Peter Gillespie
Fayetteville State University

Johannes Hattingh
Georgia State University

James Helmreich
Marist College

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University of New Orleans

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LaGuardia Community College

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University of Utara Malaysia

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Central Connecticut State University

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Winona State University

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Beijing University of Posts and
Telecommunications

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De Anza Community College

Tamsen McGinley
Santa Clara University

Ramon A. Mata-Toledo
James Madison University

Tamara Melnik
Computer Systems Institute

Osvaldo Mendez
University of Texas at El Paso

Darrell Minor
Columbus State Community College

Kathleen O'Connor
Quinsigamond Community College

Keith Olson
Utah Valley University

Dimitris Papamichail
The College of New Jersey

Yongyuth Permpoontanalarp
King Mongkut's University of
Technology, Thonburi

Galin Piatniskaia
University of Missouri, St. Louis

Shawon Rahman
University of Hawaii at Hilo

Eric Rawdon
University of St. Thomas

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Auburn University

Sukhit Singh
Texas State University, San Marcos

David Snyder
Texas State University, San Marcos

Wasin So
San Jose State University

Bogdan Suceava
California State University, Fullerton

Christopher Swanson
Ashland University

Bon Sy
Queens College

Fereja Tahir
Illinois Central College

K.A. Venkatesh
Presidency University

Matthew Walsh
Indiana-Purdue University, Fort
Wayne

Sheri Wang
University of Phoenix

Gideon Weinstein
Western Governors University

David Wilczynski
University of Southern California

James Wooland
Florida State University

I also want to thank the many students who have provided suggestions and reported errata. The students in the discrete mathematics courses I have taught at Monmouth University, both undergraduate and graduate computer science students, have helped me improve the book in many ways.

There are many people to thank at McGraw-Hill Higher Education, the publisher of this book, as well as people who did the production work at Aptara. I would also like to thank the original editor at Random House, Wayne Yuhasz, whose insights and skills helped ensure the book's success, as well as all the many other previous editors of this book.

I want to express my deep appreciation to Nora Devlin, the Product Developer who has gone far beyond her assigned duties to support the author. She has displayed many skills and virtues working to solve a wide variety of problems that have arisen in the development of this new edition.

I am also grateful to Peggy Selle, the Content Product Manager, who managed the production process. She has kept the production on track and has helped resolve many issues that have arise during this process. Thanks go to Sarita Yadav, Senior Product Manager of Aptara, and her colleagues at Aptara, who worked diligently to ensure the production quality of this edition.

I also want to express my appreciation to the many others in the Science, Engineering, and Mathematics (SEM) Division of McGraw-Hill Higher Education for their valuable support for this new edition and the associated media content, including

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- Ruth Czarnecki-Lichstein, Business Product Manager
- Megan Platt, Editorial Coordinator
- Lora Neyens and Jolynn Kilburg, Program Managers
- Lorraine Buczek, Content Licensing Specialist

Kenneth H. Rosen

Online Resources

Extensive effort has been devoted to producing valuable web resources for this book. Instructors should make a special effort to explore these resources to identify those they feel will help their students learn and explore discrete mathematics. These resources are available in the Online Learning Center, which is available to all students and instructors, and the Connect Site, designed for interactive instruction, which instructors can choose to use. To use Connect, students purchase online access for a specific time period.

0.1 The Online Learning Center

The Online Learning Center (OLC), accessible at *www.mhhe.com/rosen*, includes an *Information Center*, a *Student Site*, and an *Instructor Site*. Key features of each area are described here.

0.1.1 The Information Center

The Information Center contains basic information about the book including the expanded table of contents (including subsection heads), the preface, descriptions of the ancillaries, and a sample chapter. It also provides a link that can be used to submit errata reports and other feedback about the book.

0.1.2 Student Site

The Student Site contains a wealth of resources available for student use, including the following, tied into the text wherever the special icons displayed below are found in the text:

Extra Examples

- ***Extra Examples*** You can find a large number of additional examples on the site, covering all chapters of the book. These examples are concentrated in areas where students often ask for additional material. Although most of these examples amplify the basic concepts, more-challenging examples can also be found here. Many new extra examples have been recently added for the eighth edition. Each icon in the book corresponds to one or more extra examples on the website.

Demo

- ***Interactive Demonstration Applets*** These applets enable you to interactively explore how important algorithms work, and are tied directly to material in the text with linkages to examples and exercises. Additional resources are provided on how to use and apply these applets.

Assessment

- ***Self Assessments*** These interactive guides help you assess your understanding of 14 key concepts, providing a question bank where each question includes a brief tutorial followed by a multiple-choice question. If you select an incorrect answer, advice is provided to help you understand your error. Using these Self Assessments, you should be able to diagnose your problems and find appropriate help.

Links

- ***Web Resources Guide*** This guide provides annotated links to hundreds of external websites containing relevant material such as historical and biographical information, puzzles and problems, discussions, applets, programs, and more. These links are keyed to the text by page number.

Additional resources in the Student Site include:

- ***Exploring Discrete Mathematics*** This ancillary provides help for using a computer algebra system to do a wide range of computations in discrete mathematics. Each chapter provides a description of relevant functions in the computer algebra system and how they are used, programs to carry out computations in discrete mathematics, examples, and exercises that can be worked using this computer algebra system. Two versions, *Exploring Discrete Mathematics with Maple*™ and *Exploring Discrete Mathematics with Mathematica*™, are available.
- ***Applications of Discrete Mathematics*** This ancillary contains 24 chapters—each with its own set of exercises—presenting a wide variety of interesting and important applications covering three general areas in discrete mathematics: discrete structures, combinatorics, and graph theory. These applications are ideal for supplementing the text or for independent study.
- ***A Guide to Proof-Writing*** This guide provides additional help for writing proofs, a skill that many students find difficult to master. By reading this guide at the beginning of the course and periodically thereafter when proof writing is required, you will be rewarded as your proof-writing ability grows. (Also available in the *Student's Solutions Guide*.)
- ***Common Mistakes in Discrete Mathematics*** This guide includes a detailed list of common misconceptions that students of discrete mathematics often have and the kinds of errors they tend to make. You are encouraged to review this list from time to time to help avoid these common traps. (Also available in the *Student's Solutions Guide*.)
- ***Advice on Writing Projects*** This guide offers helpful hints and suggestions for the Writing Projects in the text, including an extensive bibliography of helpful books and articles for research, discussion of various resources available in print and online, tips on doing library research, and suggestions on how to write well. (Also available in the *Student's Solutions Guide*.)

0.1.3 Instructor Site

This part of the website provides access to all of the resources on the Student Site, as well as these resources for instructors:

- ***Suggested Syllabi*** Detailed course outlines are shown, offering suggestions for courses with different emphases and different student backgrounds and ability levels.
- ***Teaching Suggestions*** This guide contains detailed teaching suggestions for instructors, including chapter overviews for the entire text, detailed remarks on each section, and comments on the exercise sets.
- ***Printable Tests*** Printable tests are offered in TeX and Word format for every chapter, and can be customized by instructors.
- ***PowerPoint Lecture Slides and PowerPoint Figures and Tables*** An extensive collection of PowerPoint lecture notes for all chapters of the text are provided for instructor use. In addition, images of all figures and tables from the text are provided as PowerPoint slides.

0.1.4 Connect

A comprehensive online learning package has been developed in conjunction with the text. A high-level description of this site will be provided here. Interested instructors and students can find out more about Connect from McGraw-Hill Higher Education. When instructors choose to use this option, students in their classes must obtain access to Connect for this text, either by purchasing a copy of the textbook that also includes access privileges or by purchasing access only with the option of buying a loose-leaf version of the textbook.

Instructors who adopt Connect have access to a full-featured course management system. Course management capabilities are provided that allow instructors to create assignments, automatically assign and grade homework, quiz, and test questions from a bank of questions tied directly to the text, create and edit their own questions, manage course announcements and due dates, and track student progress.

Instructors can create their own assignments using Connect. They select the particular exercises from each section of the book that they want to assign. They can also assign chapters from the SmartBook version of the text, which provides an adaptive learning environment. Students have access to the interactive version of the textbook, the online homework exercises, and SmartBook.

Interactive Textbook Students have access to an easy-to-use interactive version of the textbook when they use Connect. The interactive site provides the full content of the text, as well as the many extra resources that enrich the book. The resources include extra examples, interactive demonstrations, and self-assessments.

Homework and Learning Management Solution An extensive learning management solution has been developed that instructors can use to construct homework assignments. Approximately 800 online questions are available, including questions from every section of the text. These questions are tied to the most commonly assigned exercises in the book.

These online questions have been constructed to support the same objectives as the corresponding written homework questions. This challenge has been met by stretching the capabilities of different question types supported by the Connect platform.

SmartBook Connect also provides another extended online version of the text in the McGraw-Hill SmartBook platform. The SmartBook version of the text includes a set of objectives for each chapter of the text. A collection of questions, called probes, is provided to assess student understanding of each objective. Students are directed to the appropriate part of the text to review the material needed for each of these objectives. SmartBook provides an adaptive learning environment; it selects probes for students based on their performance answering earlier probes. Instructors can assign SmartBook as assignments or can have their students use SmartBook as a learning tool.

To the Student

What is discrete mathematics? Discrete mathematics is the part of mathematics devoted to the study of discrete objects. (Here *discrete* means consisting of distinct or unconnected elements.) The kinds of problems solved using discrete mathematics include:

- How many ways are there to choose a valid password on a computer system?
- What is the probability of winning a lottery?
- Is there a link between two computers in a network?
- How can I identify spam e-mail messages?
- How can I encrypt a message so that no unintended recipient can read it?
- What is the shortest path between two cities using a transportation system?
- How can a list of integers be sorted so that the integers are in increasing order?
- How many steps are required to do such a sorting?
- How can it be proved that a sorting algorithm correctly sorts a list?
- How can a circuit that adds two integers be designed?
- How many valid Internet addresses are there?

You will learn the discrete structures and techniques needed to solve problems such as these.

More generally, discrete mathematics is used whenever objects are counted, when relationships between finite (or countable) sets are studied, and when processes involving a finite number of steps are analyzed. A key reason for the growth in the importance of discrete mathematics is that information is stored and manipulated by computing machines in a discrete fashion.

WHY STUDY DISCRETE MATHEMATICS? There are several important reasons for studying discrete mathematics. First, through this course you can develop your mathematical maturity: that is, your ability to understand and create mathematical arguments. You will not get very far in your studies in the mathematical sciences without these skills.

Second, discrete mathematics is the gateway to more advanced courses in all parts of the mathematical sciences. Discrete mathematics provides the mathematical foundations for many computer science courses, including data structures, algorithms, database theory, automata theory, formal languages, compiler theory, computer security, and operating systems. Students find these courses much more difficult when they have not had the appropriate mathematical foundations from discrete mathematics. One student sent me an e-mail message saying that she used the contents of this book in every computer science course she took!

Math courses based on the material studied in discrete mathematics include logic, set theory, number theory, linear algebra, abstract algebra, combinatorics, graph theory, and probability theory (the discrete part of the subject).

Also, discrete mathematics contains the necessary mathematical background for solving problems in operations research (including discrete optimization), chemistry, engineering, biology, and so on. In the text, we will study applications to some of these areas.

Many students find their introductory discrete mathematics course to be significantly more challenging than courses they have previously taken. One reason for this is that one of the primary goals of this course is to teach mathematical reasoning and problem solving, rather than a discrete set of skills. The exercises in this book are designed to reflect this goal. Although there are plenty of exercises in this text similar to those addressed in the examples, a large percentage of the exercises require original thought. This is intentional. The material discussed in the text provides the tools needed to solve these exercises, but your job is to successfully apply these tools using your own creativity. One of the primary goals of this course is to learn how to attack problems that may be somewhat different from any you may have previously seen. Unfortunately, learning how to solve only particular types of exercises is not sufficient for success in developing the problem-solving skills needed in subsequent courses and professional work. This text addresses many different topics, but discrete mathematics is an extremely diverse and large area of study. One of my goals as an author is to help you develop the skills needed to master the additional material you will need in your own future pursuits.

Finally, discrete mathematics is an excellent environment in which to learn how to read and write mathematical proofs. In addition to explicit material on proofs in Chapter 1 and Chapter 5, this textbook contains throughout many proofs of theorems and many exercises asking the student to prove statements. This not only deepens one's understanding of the subject matter but is also valuable preparation for more advanced courses in mathematics and theoretical computer science.

THE EXERCISES I would like to offer some advice about how you can best learn discrete mathematics (and other subjects in the mathematical and computing sciences). You will learn the most by actively working exercises. I suggest that you solve as many as you possibly can. After working the exercises your instructor has assigned, I encourage you to solve additional exercises such as those in the exercise sets following each section of the text and in the supplementary exercises at the end of each chapter. (Note the key explaining the markings preceding exercises.)

Key to the Exercises

no marking	A routine exercise
*	A difficult exercise
**	An extremely challenging exercise
☞	An exercise containing a result used in the book (Table 1 on the following page shows where these exercises are used.)
(*Requires calculus*)	An exercise whose solution requires the use of limits or concepts from differential or integral calculus

The best approach is to try exercises yourself before you consult the answer section at the end of this book. Note that the odd-numbered exercise answers provided in the text are answers only and not full solutions; in particular, the reasoning required to obtain answers is omitted in these answers. The *Student's Solutions Guide*, available separately, provides complete, worked solutions to all odd-numbered exercises in this text. When you hit an impasse trying to solve an odd-numbered exercise, I suggest you consult the *Student's Solutions Guide* and look for some guidance as to how to solve the problem. The more work you do yourself rather than passively reading or copying solutions, the more you will learn. The answers and solutions to the even-numbered exercises are intentionally not available from the publisher; ask your instructor if you have trouble with these.

WEB RESOURCES All users of the book are able to access the online resources accessible via the Online Learning Center (OLC) for the book. You will find many Extra Examples designed to clarify key concepts, Self Assessments for gauging how well you understand core topics, Interactive Demonstrations that explore key algorithms and other concepts, a Web Resources Guide containing an extensive selection of links to external sites relevant to the world of discrete mathematics, extra explanations and practice to help you master core concepts, added instruction on writing proofs and on avoiding common mistakes in discrete mathematics, in-depth discussions of important applications, and guidance on utilizing Maple™ and

TABLE 1 Hand-Icon Exercises and Where They Are Used

Section	*Exercise*	*Section Where Used*	*Pages Where Used*
1.1	42	1.3	33
1.1	43	1.3	33
1.3	11	1.6	76
1.3	12	1.6	74, 76
1.3	19	1.6	76
1.3	34	1.6	76, 78
1.3	46	12.2	856
1.7	18	1.7	86
2.3	74	2.3	144
2.3	81	2.5	170
2.5	15	2.5	174
2.5	16	2.5	173
3.1	45	3.1	197
3.2	74	11.2	797
4.3	37	4.1	253
4.4	2	4.6	318
4.4	44	7.2	489
6.4	21	7.2	491
6.4	25	7.4	480
7.2	15	7.2	491
9.1	26	9.4	629
10.4	59	11.1	782
11.1	15	11.1	786
11.1	30	11.1	791
11.1	48	11.2	798
12.1	12	12.3	861
A.2	4	8.3	531

Mathematica™ software to explore the computational aspects of discrete mathematics. Places in the text where these additional online resources are available are identified in the margins by special icons. For more details on these and other online resources, see the description of the companion website immediately preceding this “To the Student” message.

THE VALUE OF THIS BOOK My intention is to make your substantial investment in this text an excellent value. The book, the associated ancillaries, and companion website have taken many years of effort to develop and refine. I am confident that most of you will find that the text and associated materials will help you master discrete mathematics, just as so many previous students have. Even though it is likely that you will not cover some chapters in your current course, you should find it helpful—as many other students have—to read the relevant sections of the book as you take additional courses. Most of you will return to this book as a useful tool throughout your future studies, especially for those of you who continue in computer science, mathematics, and engineering. I have designed this book to be a gateway for future studies and explorations, and to be comprehensive reference, and I wish you luck as you begin your journey.

Kenneth H. Rosen

CHAPTER

1 The Foundations: Logic and Proofs

The rules of logic specify the meaning of mathematical statements. For instance, these rules help us understand and reason with statements such as "There exists an integer that is not the sum of two squares" and "For every positive integer n, the sum of the positive integers not exceeding n is $n(n+1)/2$." Logic is the basis of all mathematical reasoning, and of all automated reasoning. It has practical applications to the design of computing machines, to the specification of systems, to artificial intelligence, to computer programming, to programming languages, and to other areas of computer science, as well as to many other fields of study.

To understand mathematics, we must understand what makes up a correct mathematical argument, that is, a proof. Once we prove a mathematical statement is true, we call it a theorem. A collection of theorems on a topic organize what we know about this topic. To learn a mathematical topic, a person needs to actively construct mathematical arguments on this topic, and not just read exposition. Moreover, knowing the proof of a theorem often makes it possible to modify the result to fit new situations.

Everyone knows that proofs are important throughout mathematics, but many people find it surprising how important proofs are in computer science. In fact, proofs are used to verify that computer programs produce the correct output for all possible input values, to show that algorithms always produce the correct result, to establish the security of a system, and to create artificial intelligence. Furthermore, automated reasoning systems have been created to allow computers to construct their own proofs.

In this chapter, we will explain what makes up a correct mathematical argument and introduce tools to construct these arguments. We will develop an arsenal of different proof methods that will enable us to prove many different types of results. After introducing many different methods of proof, we will introduce several strategies for constructing proofs. We will introduce the notion of a conjecture and explain the process of developing mathematics by studying conjectures.

1.1 Propositional Logic

1.1.1 Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. Because a major goal of this book is to teach the reader how to understand and how to construct correct mathematical arguments, we begin our study of discrete mathematics with an introduction to logic.

Besides the importance of logic in understanding mathematical reasoning, logic has numerous applications to computer science. These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways. Furthermore, software systems have been developed for constructing some, but not all, types of proofs automatically. We will discuss these applications of logic in this and later chapters.

1.1.2 Propositions

Our discussion begins with an introduction to the basic building blocks of logic—propositions. A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

EXAMPLE 1 All the following declarative sentences are propositions.

Extra Examples

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. $1 + 1 = 2$.
4. $2 + 2 = 3$.

Propositions 1 and 3 are true, whereas 2 and 4 are false. ◀

Some sentences that are not propositions are given in Example 2.

EXAMPLE 2 Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables. We will also discuss other ways to turn sentences such as these into propositions in Section 1.4. ◀

We use letters to denote **propositional variables** (or **sentential variables**), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are $p, q, r, s, \dots$. The **truth value** of a proposition

Links

Source: National Library of Medicine

ARISTOTLE (384 B.C.E.–322 B.C.E.) Aristotle was born in Stagirus (Stagira) in northern Greece. His father was the personal physician of the King of Macedonia. Because his father died when Aristotle was young, Aristotle could not follow the custom of following his father's profession. Aristotle became an orphan at a young age when his mother also died. His guardian who raised him taught him poetry, rhetoric, and Greek. At the age of 17, his guardian sent him to Athens to further his education. Aristotle joined Plato's Academy, where for 20 years he attended Plato's lectures, later presenting his own lectures on rhetoric. When Plato died in 347 B.C.E., Aristotle was not chosen to succeed him because his views differed too much from those of Plato. Instead, Aristotle joined the court of King Hermeas where he remained for three years, and married the niece of the King. When the Persians defeated Hermeas, Aristotle moved to Mytilene and, at the invitation of King Philip of Macedonia, he tutored Alexander, Philip's son, who later became Alexander the Great. Aristotle tutored Alexander for five years and after the death of King Philip, he returned to Athens and set up his own school, called the Lyceum.

Aristotle's followers were called the peripatetics, which means "to walk about," because Aristotle often walked around as he discussed philosophical questions. Aristotle taught at the Lyceum for 13 years where he lectured to his advanced students in the morning and gave popular lectures to a broad audience in the evening. When Alexander the Great died in 323 B.C.E., a backlash against anything related to Alexander led to trumped-up charges of impiety against Aristotle. Aristotle fled to Chalcis to avoid prosecution. He only lived one year in Chalcis, dying of a stomach ailment in 322 B.C.E.

Aristotle wrote three types of works: those written for a popular audience, compilations of scientific facts, and systematic treatises. The systematic treatises included works on logic, philosophy, psychology, physics, and natural history. Aristotle's writings were preserved by a student and were hidden in a vault where a wealthy book collector discovered them about 200 years later. They were taken to Rome, where they were studied by scholars and issued in new editions, preserving them for posterity.

is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition. Propositions that cannot be expressed in terms of simpler propositions are called **atomic propositions**.

The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**. It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.

Links

We now turn our attention to methods for producing new propositions from those that we already have. These methods were discussed by the English mathematician George Boole in 1854 in his book *The Laws of Thought.* Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using **logical operators**.

Definition 1

Let p be a proposition. The *negation of* p, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement

"It is not the case that p."

The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Remark: The notation for the negation operator is not standardized. Although $\neg p$ and $\overline{p}$ are the most common notations used in mathematics to express the negation of p, other notations you might see are $\sim p$, $-p$, p', Np, and $!p$.

EXAMPLE 3 Find the negation of the proposition

"Michael's PC runs Linux"

Extra Examples

and express this in simple English.

Solution: The negation is

"It is not the case that Michael's PC runs Linux."

This negation can be more simply expressed as

"Michael's PC does not run Linux." ◀

EXAMPLE 4 Find the negation of the proposition

"Vandana's smartphone has at least 32 GB of memory"

and express this in simple English.

Solution: The negation is

"It is not the case that Vandana's smartphone has at least 32 GB of memory."

This negation can also be expressed as

"Vandana's smartphone does not have at least 32 GB of memory"

or even more simply as

"Vandana's smartphone has less than 32 GB of memory." ◀

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

Table 1 displays the **truth table** for the negation of a proposition p. This table has a row for each of the two possible truth values of p. Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

The negation of a proposition can also be considered the result of the operation of the **negation operator** on a proposition. The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called **connectives**.

Definition 2 Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Table 2 displays the truth table of $p \wedge q$. This table has a row for each of the four possible combinations of truth values of p and q. The four rows correspond to the pairs of truth values TT, TF, FT, and FF, where the first truth value in the pair is the truth value of p and the second truth value is the truth value of q.

Note that in logic the word "but" sometimes is used instead of "and" in a conjunction. For example, the statement "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining." (In natural language, there is a subtle difference in meaning between "and" and "but"; we will not be concerned with this nuance here.)

EXAMPLE 5 Find the conjunction of the propositions p and q where p is the proposition "Rebecca's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Rebecca's PC runs faster than 1 GHz."

Solution: The conjunction of these propositions, $p \wedge q$, is the proposition "Rebecca's PC has more than 16 GB free hard disk space, and the processor in Rebecca's PC runs faster than 1 GHz." This conjunction can be expressed more simply as "Rebecca's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz." For this conjunction to be true, both conditions given must be true. It is false when one or both of these conditions are false. ◀

Definition 3 Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Table 3 displays the truth table for $p \vee q$.

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The use of the connective *or* in a disjunction corresponds to one of the two ways the word *or* is used in English, namely, as an **inclusive or**. A disjunction is true when at least one of the two propositions is true. That is, $p \vee q$ is true when both p and q are true or when exactly one of p and q is true.

EXAMPLE 6 Translate the statement "Students who have taken calculus or introductory computer science can take this class" in a statement in propositional logic using the propositions p: "A student who has taken calculus can take this class" and q: "A student who has taken introductory computer science can take this class."

Solution: We assume that this statement means that students who have taken both calculus and introductory computer science can take the class, as well as the students who have taken only one of the two subjects. Hence, this statement can be expressed as $p \vee q$, the inclusive or, or disjunction, of p and q. ◀

EXAMPLE 7 What is the disjunction of the propositions p and q, where p and q are the same propositions as in Example 5?

Extra Examples

Solution: The disjunction of p and q, $p \vee q$, is the proposition

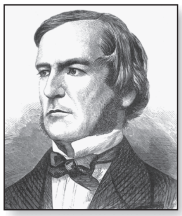
> "Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebecca's PC runs faster than 1 GHz."

This proposition is true when Rebecca's PC has at least 16 GB free hard disk space, when the PC's processor runs faster than 1 GHz, and when both conditions are true. It is false when both of these conditions are false, that is, when Rebecca's PC has less than 16 GB free hard disk space and the processor in her PC runs at 1 GHz or slower. ◀

Besides its use in disjunctions, the connective *or* is also used to express an *exclusive or*. Unlike the disjunction of two propositions p and q, the exclusive or of these two propositions is true when exactly one of p and q is true; it is false when both p and q are true (and when both are false).

Definition 4 Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$ (or p XOR q), is the proposition that is true when exactly one of p and q is true and is false otherwise.

Links



Source: Library of Congress Washington, D.C. 20540 USA [LC-USZ62-61664]

GEORGE BOOLE (1815–1864) George Boole, the son of a cobbler, was born in Lincoln, England, in November 1815. Because of his family's difficult financial situation, Boole struggled to educate himself while supporting his family. Nevertheless, he became one of the most important mathematicians of the 1800s. Although he considered a career as a clergyman, he decided instead to go into teaching, and soon afterward opened a school of his own. In his preparation for teaching mathematics, Boole—unsatisfied with textbooks of his day—decided to read the works of the great mathematicians. While reading papers of the great French mathematician Lagrange, Boole made discoveries in the calculus of variations, the branch of analysis dealing with finding curves and surfaces by optimizing certain parameters.

In 1848 Boole published *The Mathematical Analysis of Logic*, the first of his contributions to symbolic logic. In 1849 he was appointed professor of mathematics at Queen's College in Cork, Ireland. In 1854 he published *The Laws of Thought*, his most famous work. In this book, Boole introduced what is now called *Boolean algebra* in his honor. Boole wrote textbooks on differential equations and on difference equations that were used in Great Britain until the end of the nineteenth century. Boole married in 1855; his wife was the niece of the professor of Greek at Queen's College. In 1864 Boole died from pneumonia, which he contracted as a result of keeping a lecture engagement even though he was soaking wet from a rainstorm.

The truth table for the exclusive or of two propositions is displayed in Table 4.

EXAMPLE 8 Let p and q be the propositions that state "A student can have a salad with dinner" and "A student can have soup with dinner," respectively. What is $p \oplus q$, the exclusive or of p and q?

Solution: The exclusive or of p and q is the statement that is true when exactly one of p and q is true. That is, $p \oplus q$ is the statement "A student can have soup or salad, but not both, with dinner." Note that this is often stated as "A student can have soup or a salad with dinner," without explicitly stating that taking both is not permitted. ◀

EXAMPLE 9 Express the statement "I will use all my savings to travel to Europe or to buy an electric car" in propositional logic using the statement p: "I will use all my savings to travel to Europe" and the statement q: "I will use all my savings to buy an electric car."

Solution: To translate this statement, we first note that the or in this statement must be an exclusive or because this student can either use all his or her savings to travel to Europe or use all these savings to buy an electric car, but cannot both go to Europe and buy an electric car. (This is clear because either option requires all his savings.) Hence, this statement can be expressed as $p \oplus q$. ◀

1.1.3 Conditional Statements

We will discuss several other important ways in which propositions can be combined.

Definition 5 Let p and q be propositions. The *conditional statement* $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

Assessment

The statement $p \to q$ is called a conditional statement because $p \to q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.

The truth table for the conditional statement $p \to q$ is shown in Table 5. Note that the statement $p \to q$ is true when both p and q are true and when p is false (no matter what truth value q has).

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5 The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \to q$. You will encounter most if not all of the following ways to express this conditional statement:

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless $\neg p$"	"q provided that p"

A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is

"If I am elected, then I will lower taxes."

If the politician is elected, voters would expect this politician to lower taxes. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes. It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true but q is false in $p \to q$.

Similarly, consider a statement that a professor might make:

"If you get 100% on the final, then you will get an A."

If you manage to get 100% on the final, then you would expect to receive an A. If you do not get 100%, you may or may not receive an A depending on other factors. However, if you do get 100%, but the professor does not give you an A, you will feel cheated.

Remark: Because some of the different ways to express the implication p implies q can be confusing, we will provide some extra guidance. To remember that "p only if q" expresses the same thing as "if p, then q," note that "p only if q" says that p cannot be true when q is not true. That is, the statement is false if p is true, but q is false. When p is false, q may be either true or false, because the statement says nothing about the truth value of q.

For example, suppose your professor tells you

"You can receive an A in the course only if your score on the final is at least 90%."

Then, if you receive an A in the course, then you know that your score on the final is at least 90%. If you do not receive an A, you may or may not have scored at least 90% on the final. Be careful not to use "q only if p" to express $p \to q$ because this is incorrect. The word "only" plays an essential role here. To see this, note that the truth values of "q only if p" and $p \to q$ are different when p and q have different truth values. To see why "q is necessary for p" is equivalent to "if p, then q," observe that "q is necessary for p" means that p cannot be true unless q is true, or that if q is false, then p is false. This is the same as saying that if p is true, then q is true. To see why "p is sufficient for q" is equivalent to "if p, then q," note that "p is sufficient for q" means if p is true, it must be the case that q is also true. This is the same as saying that if p is true, then q is also true.

You might have trouble understanding how "unless" is used in conditional statements unless you read this paragraph carefully.

To remember that "q unless $\neg p$" expresses the same conditional statement as "if p, then q," note that "q unless $\neg p$" means that if $\neg p$ is false, then q must be true. That is, the statement "q unless $\neg p$" is false when p is true but q is false, but it is true otherwise. Consequently, "q unless $\neg p$" and $p \to q$ always have the same truth value.

We illustrate the translation between conditional statements and English statements in Example 10.

EXAMPLE 10 Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \to q$ as a statement in English.

Extra Examples

Solution: From the definition of conditional statements, we see that when p is the statement "Maria learns discrete mathematics" and q is the statement "Maria will find a good job," $p \to q$ represents the statement

"If Maria learns discrete mathematics, then she will find a good job."

There are many other ways to express this conditional statement in English. Among the most natural of these are

"Maria will find a good job when she learns discrete mathematics."

"For Maria to get a good job, it is sufficient for her to learn discrete mathematics."

and

"Maria will find a good job unless she does not learn discrete mathematics." ◀

Note that the way we have defined conditional statements is more general than the meaning attached to such statements in the English language. For instance, the conditional statement in Example 10 and the statement

"If it is sunny, then we will go to the beach"

are statements used in normal language where there is a relationship between the hypothesis and the conclusion. Further, the first of these statements is true unless Maria learns discrete mathematics, but she does not get a good job, and the second is true unless it is indeed sunny, but we do not go to the beach. On the other hand, the statement

"If Juan has a smartphone, then $2 + 3 = 5$"

is true from the definition of a conditional statement, because its conclusion is true. (The truth value of the hypothesis does not matter then.) The conditional statement

"If Juan has a smartphone, then $2 + 3 = 6$"

is true if Juan does not have a smartphone, even though $2 + 3 = 6$ is false. We would not use these last two conditional statements in natural language (except perhaps in sarcasm), because there is no relationship between the hypothesis and the conclusion in either statement. In mathematical reasoning, we consider conditional statements of a more general sort than we use in English. The mathematical concept of a conditional statement is independent of a cause-and-effect relationship between hypothesis and conclusion. Our definition of a conditional statement specifies its truth values; it is not based on English usage. Propositional language is an artificial language; we only parallel English usage to make it easy to use and remember.

The if-then construction used in many programming languages is different from that used in logic. Most programming languages contain statements such as **if** p **then** S, where p is a proposition and S is a program segment (one or more statements to be executed). (Although this looks as if it might be a conditional statement, S is not a proposition, but rather is a set of executable instructions.) When execution of a program encounters such a statement, S is executed if p is true, but S is not executed if p is false, as illustrated in Example 11.

EXAMPLE 11 What is the value of the variable x after the statement

$$\textbf{if } 2 + 2 = 4 \textbf{ then } x := x + 1$$

if $x = 0$ before this statement is encountered? (The symbol := stands for assignment. The statement $x := x + 1$ means the assignment of the value of $x + 1$ to x.)

Solution: Because $2 + 2 = 4$ is true, the assignment statement $x := x + 1$ is executed. Hence, x has the value $0 + 1 = 1$ after this statement is encountered. ◀

CONVERSE, CONTRAPOSITIVE, AND INVERSE We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \to p$ is called the **converse** of $p \to q$. The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$. The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$. We will see that of these three conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$.

We first show that the contrapositive, $\neg q \to \neg p$, of a conditional statement $p \to q$ always has the same truth value as $p \to q$. To see this, note that the contrapositive is false only when $\neg p$ is false and $\neg q$ is true, that is, only when p is true and q is false. We now show that neither the converse, $q \to p$, nor the inverse, $\neg p \to \neg q$, has the same truth value as $p \to q$ for all possible truth values of p and q. Note that when p is true and q is false, the original conditional statement is false, but the converse and the inverse are both true.

Remember that the contrapositive, but neither the converse or inverse, of a conditional statement is equivalent to it.

When two compound propositions always have the same truth values, regardless of the truth values of its propositional variables, we call them **equivalent**. Hence, a conditional statement and its contrapositive are equivalent. The converse and the inverse of a conditional statement are also equivalent, as the reader can verify, but neither is equivalent to the original conditional statement. (We will study equivalent propositions in Section 1.3.) Take note that one of the most common logical errors is to assume that the converse or the inverse of a conditional statement is equivalent to this conditional statement.

We illustrate the use of conditional statements in Example 12.

EXAMPLE 12 Find the contrapositive, the converse, and the inverse of the conditional statement

Extra Examples

"The home team wins whenever it is raining."

Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as

"If it is raining, then the home team wins."

Consequently, the contrapositive of this conditional statement is

"If the home team does not win, then it is not raining."

The converse is

"If the home team wins, then it is raining."

The inverse is

"If it is not raining, then the home team does not win."

Only the contrapositive is equivalent to the original statement. ◀